The Cartesian Plane

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After completing this topic you should be able to:

A1  Plot and locate points on the Cartesian plane
A2  Differentiate between dependent and independent variables in relevant contexts
A3  Graph lines from a linear equation using a table of values
A4  Find the gradient between pairs of points plotted on the Cartesian plane
A6  Recognise and model simple real life situations involving linear relationships by drawing graphs
A7  Explain the meaning in context of the intercepts (x & y) in linear models
A8  Describe limitations that may exist in linear models
A9  Solve mathematical problems graphically resulting from linear models

B1  Use the Cartesian plane to represent and analyse linear relationships
B2  Determine the equation of a straight line drawn in the Cartesian Plane
B3  Determine the equation of the line of "best fit" for a set of empirical data
B4  Solve pairs of simultaneous linear equations graphically.
B5  Solve practical linear relationship problems
Pre-Test

Use this test to self-assess your strengths and weaknesses in this topic.
Solutions are provided at the end of the booklet.
The references, provided for each question, direct you to the appropriate section of the booklet for help where needed.

QUESTION 1 (Ref. The Cartesian Plane & Coordinates)

(a) The diagram shows part of the Cartesian Plane.
Plot and label each of the following points.
   (i) P(3, 4) (ii) Q(−2, 1)
   (iii) R(0, −3) (iv) S(1, −2)

(b) Write down the coordinates for each point shown in the following diagram.

(c) For the points A and B in the diagram above:
   (i) The abscissa of A is ........ (ii) The ordinate of B is ......

4
QUESTION 2 (Ref. Graphs of Linear Functions)

(a) Copy and complete the table of values for the equation  \( y = 3x + 2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Using the values in your table to represent coordinates, draw a diagram of the Cartesian Plane showing the graph of the linear function \( y = 3x + 2 \).

QUESTION 3 (Ref. The gradient of a line & its vertical intercept.)

(a) What is the gradient and the \( y \)-intercept of the graph drawn in question 2?

(b) Without drawing graphs, write down the gradient of:

   (i) \( y = 3x - 1 \)
   (ii) \( y = -x + 5 \)

(c) Without drawing graphs, write down the intercept on the \( y \)-axis of:

   (i) \( y = x - 2 \)
   (ii) \( y = -4x + 2 \)

(d) Without using a table of values draw a diagram showing the graphs of the functions:

   (i) \( y = x + 2 \)
   (ii) \( y = -3x + 1 \)

QUESTION 4 (Ref. Linear Models)

(a) The cost of producing scones is given by the function \( C = 300 + 2x \) where \( C \) is the cost in dollars and \( x \) is the number of scones.

   (i) Use the equation to find the cost of producing 50 scones.

   (ii) Draw the graph of the function \( C = 20 + 2x \) for values of \( x \) from 0 to 10 with the dependent variable \( C \) on the vertical axis.

   (iii) Write down the vertical intercept and interpret this value.

   (iv) Find the gradient of the line and interpret this gradient in the context of the question.

(b) The outstanding amount on a loan is given by \( A = 400 - 50x \) where \( A \) is the outstanding amount in dollars and \( x \) is the number of repayments.

   (i) Find the outstanding amount after 5 repayments.

   (ii) Draw the graph of the function \( A = 400 - 50x \) for values of \( x \) from 0 to 10.

   (iii) Find the gradient of the line and describe the significance of this value.

   (iv) For what values of \( x \) does the model have practical meaning?
QUESTION 5 (Ref. Lines of Best-Fit)

The table shows the results of an experiment in which the temperature of a room was measured over a period of time.

<table>
<thead>
<tr>
<th>t (min)</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>8</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>T (°C)</td>
<td>7</td>
<td>9</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>18</td>
</tr>
</tbody>
</table>

(a) Which of the variables is the independent variable in this case?
(b) Draw a scatter diagram showing the six data pairs in the table as coordinates in the Cartesian Plane.
(c) Draw a linear line of best-fit on your diagram and find the vertical intercept.
(d) Find the gradient and interpret this value.
(e) Use your answers to (c) and (d) to write down the equation of the line of best-fit.
(d) Use the equation of the line to predict the temperature when \( t = 30 \) min and comment on the reliability of this prediction.

QUESTION 6 (Ref. Spreadsheets)

The data in the following table show the heights and ages of five children.

<table>
<thead>
<tr>
<th>h (cm)</th>
<th>95</th>
<th>110</th>
<th>130</th>
<th>140</th>
<th>156</th>
</tr>
</thead>
<tbody>
<tr>
<td>x (years)</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>

(a) Which of the variables is the dependent variable in this case?
(b) Enter the data into a spreadsheet.
(c) Obtain a scatter diagram for the data.
(d) Insert the linear line of best-fit and find its equation.
(e) Use the equation of the line of best-fit to estimate the height of a 9 year old.
(f) Use the equation to describe the rate of growth in centimetres per year.
(g) Label the diagram and insert into a Word document. Print a copy.

QUESTION 7 (Ref. Simultaneous Equations)

(i) On the same diagram draw the graphs of the linear functions \( y = 3x - 5 \) and \( y = 4 - x \).
(ii) Write down the coordinates of the point where the two graphs intersect.
(iii) Hence write down the solutions to the simultaneous equations:

\[
\begin{align*}
y &= 3x - 5 \\
y &= 4 - x
\end{align*}
\]
A fundamental tool used in this unit is the Cartesian Plane devised by the French philosopher, Renee Descartes in the seventeenth century. He uses two perpendicular number lines called axes to represent two variables.

This diagram shows part of the Cartesian Plane for the variables $x$ and $y$.

The axes provide the framework for a grid that enables us to locate any point in two-dimensional space, relative to the two axes.

For example, point $P$ is located at the intersection of the lines through $x = 3$ and $y = 2$.

These $x$ and $y$ values are called the coordinates of the point $P$. The coordinates are written inside parentheses separated by a comma with the $x$ value, (called the abscissa) first, and the $y$ value, (called the ordinate) second.

So the notation $P(3, 2)$ indicates a point called $P$ with abscissa 3 and ordinate 2 as illustrated in the diagram.

Clearly any point can be located using this system.
- Points below the $x$-axis have negative ordinates. For example: $Q(3, -2)$.
- Points to the left of the $y$-axis have negative abscissa. For example: $R(-1, 3)$.
- Points below the $x$-axis and to the left of the $y$-axis have both negative abscissa and ordinates. For example: $S(-4, -1)$.

Indicate points $Q$, $R$ and $S$ on the diagram above, and then consolidate your skill in plotting and labelling points in the Cartesian Plane by working the Activity 1.
**Activity 1**

**QUESTION 1**
Point P in the diagram has coordinates (4,2).
Write down the coordinates of points A, B, C and D.

**QUESTION 2**
Mark and label the following points in the Cartesian Plane:
- F (5, 4)
- G (-1, -3)
- H (0, -2)
- I (4, -3.5)
- J (0, -3)

**QUESTION 3**
(a) Write down the ordinate of the point S (8, -2).
(b) Write down the abscissa of the point T (-1, 3).
(c) On a diagram plot and label the points: A(2, 1), B(2, 0), C(2, -1), D(2, 4).
(d) On another diagram plot the points: E(1, 3), F(-1, 3), G(0, 3), H(-2, 3).
(e) On another diagram plot the points: A(-3, -2), B(-1, 0), C(0, 1), D(2, 3).
(f) Write down your observations of your answers to questions (c), (d) and (e).
INTRODUCTION
The diagram shows the solution to question 3(e) in Activity 1.

You can see that the points are not just a random selection. They form a pattern. In particular this pattern of points forms a straight line.

Patterns of points arise when there is a connection or relationship between the $x$ and $y$ variables. For example, the points here might have been obtained from a scientific experiment. Discovering the relationship between the $x$ and $y$ variables is a useful and important skill to have.

EQUATIONS
Did you spot the relationship between the two variables, $x$ and $y$, in the diagram above? In each case the ordinate of the point is one more than the abscissa.

This description of the relationship can best be described using an equation which looks like this:

$$y = x + 1$$

Notice that if you take another $x$-value, such as $x = 1.5$, and substitute into the equation, you obtain a $y$-value (ordinate) of $2.5$ – and this point also lies on the same straight line through points A, B, C, D and E.

Since the points are in a straight line the relationship between $x$ and $y$ is called LINEAR.
DRAWING A LINEAR GRAPH FROM ITS EQUATION

There are several methods you can use to draw the graph of a linear function. Here you will use a table of values to find a sample of points for a given relation or equation.

To illustrate this method, consider the next worked example.

Worked Examples

EXAMPLE 1

Draw the graph of the function $y = 2x - 3$.

SOLUTION

STEP 1: Draw the frame for a table of values like this.

<table>
<thead>
<tr>
<th>$x$</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

STEP 2: Enter some $x$-values of your choice

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

STEP 3: Calculate the $y$-values for each of the selected $x$-values.

Do this by substituting into the equation of the function, $y = 2x - 3$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$-9$</td>
<td>$-7$</td>
<td>$-5$</td>
<td>$-3$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$3$</td>
</tr>
</tbody>
</table>
STEP 4: Draw the Cartesian Plane and use the pairs of values from the table as coordinates.

STEP 5: Draw a straight line through the points and label the graph with its equation.
Activity 2

**QUESTION 1**
(a) Complete this table to find a sample of coordinates for the function $y = 3x - 1$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Plot the points found in (a) on the Cartesian plane.
(c) Draw a graph through the points in (b) and label the graph with its equation.

**QUESTION 2**
(a) Complete a table to show a sample of coordinates for the function $y = 2x - 1$.
(b) Plot the points found in (a) on the same diagram used for question 1.
(c) Draw and label the graph of the function $y = 2x - 1$.
(d) Describe the difference between the graphs of $y = 3x - 1$ and $y = 2x - 1$.

**QUESTION 3**
(a) Complete a table to show a sample of coordinates for the function $y = 2x + 1$.
(b) Draw a diagram of the Cartesian Plane and plot the points found in (a).
(c) Hence, draw and label the graph of the function $y = 2x + 1$.

**QUESTION 4**
(a) Complete a table to show a sample of coordinates for the function $y = 2x - 3$.
(b) Plot the points found in (a) on the same diagram used for question 3.
(c) Draw and label the graph of the function $y = 2x - 3$.
(d) Describe the difference between the graphs of $y = 2x - 3$ and $y = 2x + 1$. 
The Gradient of a Line

INTRODUCTION
As you drew the graphs of linear relations in the previous activity, you probably noticed that graphs differ in their steepness.

This diagram shows the graphs of two linear functions, \( y = 2x - 1 \) and \( y = 3x - 1 \).

Clearly the graph of the function \( y = 3x - 1 \) is steeper than the graph of the function \( y = 2x - 1 \).

The measure of a line’s steepness is called its gradient.

FINDING GRADIENTS
The gradient of a straight line is the ratio of its rise to its run.

This is illustrated here for the graph of the function \( y = 3x - 1 \).

\[
\text{Gradient} = \frac{\text{rise}}{\text{run}} = \frac{AB}{BC} = \frac{3}{1} = 3
\]

The line \( y = 3x - 1 \) has a gradient of 3. For every one unit of increase in the \( x \) variable the \( y \) variable increases by 3 units.
A FORMULA FOR CALCULATING THE GRADIENT OF A LINE

This diagram shows the points $P(6, 4)$ and $Q(1, 2)$.

The gradient of the line $PQ$ is a measure of the steepness of the line and is defined by the ratio of the line’s rise, $PR$, to its run, $RQ$.

It is important to note:

- $PR$ is the difference between the \textbf{ordinates} of points $P$ and $Q$. (The difference between the $y$-values, $4 - 2$)
- $QR$ is the difference between the \textbf{abscissae} of $P$ and $Q$. (The difference between the $x$-values, $6 - 1$)

We now have:

$$\text{Gradient} = \frac{PR}{QR} = \frac{\text{Difference between } y\text{-values}}{\text{Difference between } x\text{-values}} = \frac{4 - 2}{6 - 1} = \frac{2}{5}$$

To generalise on this result:

If one point on the line has coordinates $(x_1, y_1)$ and a second point has coordinates $(x_2, y_2)$, the gradient of the straight line through these two points is given by:

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

The gradient is often denoted by the pronumeral $m$, so we write:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

A FORMULA WORTH REMEMBERING
EXAMPLE 1
Find the gradient of the line joining the points:
\[ P(2, 1) \text{ and } Q(6, 4) \]

METHOD

1. Decide which point is to be \((x_1, y_1)\) and
   which \((x_2, y_2)\) .............................. Let \( P = (x_1, y_1) \) so \( x_1 = 2 \) and \( y_1 = 1 \)
   Let \( Q = (x_2, y_2) \) so \( x_2 = 6 \) and \( y_2 = 4 \)

2. Quote the gradient formula ..................... \[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

3. Substitute the ordinates and abscissae, as
   selected in 1 ................................. \[ m = \frac{4 - 1}{6 - 2} \]
   and simplify.................................. \[ m = \frac{3}{4} \]

NOTES...

1. After finding the gradient of the line check that your answer is reasonable by comparing it with the diagram.
2. Always show your method and present your work as shown above.
3. It doesn’t matter which point you nominate as \((x_1, y_1)\) and which you call \((x_2, y_2)\), in step 1 so long as you are consistent with your substitution.
4. Do take extra care when the coordinates contain negative ordinates and/or abscissae.

The need for care with coordinates containing negative values is illustrated in the next worked example.
EXAMPLE 2
Find the gradient of the line joining the points:

\[ R \{-4, 1\} \text{ and } S \{-3, -2\} \]

METHOD

① Decide which point is to be \((x_1, y_1)\) and which \((x_2, y_2)\).

Let \( R = (x_1, y_1) \) so \( x_1 = -4 \) and \( y_1 = 1 \)
Let \( S = (x_2, y_2) \) so \( x_2 = -3 \) and \( y_2 = -2 \)

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

② Quote the gradient formula.

\[ m = \frac{-2 - 1}{-3 - (-4)} \]

\[ = \frac{-3}{1} \]

\[ = -3 \]

③ Substitute the ordinates and abscissa, as selected in ①.

and simplify.

CARE NEEDED HERE!

NOTES...

1. This second worked example illustrates the care needed when dealing with coordinates containing negative ordinates and/or abscissae. Use your calculator to check your computations.

2. The negative value of the gradient indicates the direction of slope. Notice that the line RS slopes **down** to the right while line PQ in worked example 1, which has a positive gradient, slopes **up** to the right.

3. It doesn’t matter which point you nominate as \((x_1, y_1)\) and which you call \((x_2, y_2)\), in step ① so long as you are consistent with your substitution.

You should verify this for yourself by repeating this example with \( S = (x_1, y_1) \) and \( R = (x_2, y_2) \)
QUESTION 1
Write down the formula for the gradient of the line joining the points \( P(x_1, y_1) \) and \( Q(x_2, y_2) \).

QUESTION 2
Find the gradient of the line joining the points \( P(5, 1) \) and \( Q(0, 4) \).

QUESTION 3
Draw a diagram to show the line segment AB where, A has coordinates \((-2, 3)\) and B has coordinates \((1, -3)\). Find the gradient of AB.

QUESTION 4
(a) For the points shown in the diagram, find the gradient of:
(i) CP
(ii) AD
(iii) BD
(b) Without calculating the gradient describe each of the following as having a positive or negative gradient:
(i) CD
(ii) BP

QUESTION 5
(a) Without drawing a diagram, determine which of the following lines slope up to the right (ie. have a positive slope), and which slope down to the right (ie. have a negative gradient).
   - Line FG where F = \((-1, 3)\) and G = \((-2, -1)\).
   - Line PQ where P = \((5, -2)\) and Q = \((3, 2)\).
   - Line ED where E = \((0, 1)\) and D = \((-3, -6)\)
(b) Show that the line segment through \((2, 3)\) and \((-1, 3)\) is horizontal.
(c) Show that the line through \((2, 3)\) and \((2, 5)\) is vertical.
QUESTION 1
Using a table of values find four points that lie on the line \( y = 3x - 2 \).

(i) Draw a diagram to show the points and hence draw the graph of the linear relation \( y = 3x - 2 \).

(ii) Using any two points on the graph find its gradient.

(iii) Extend the graph so it intersects the \( y \)-axis and write down this point of intersection.

QUESTION 2
Using a table of values find four points that lie on the line \( y = -2x + 1 \).

(i) Draw a diagram to show the points and hence draw the graph of the linear relation \( y = -2x + 1 \).

(ii) Using any two points on the graph find its gradient.

(iii) Extend the graph so it intersects the \( y \)-axis and write down this point of intersection.

QUESTION 3
(a) Repeat the method shown in questions 1 and 2 for the relation \( y = \frac{1}{2}x + 2 \)

(b) Repeat the method shown in questions 1 and 2 for the relation \( y = -\frac{2}{3}x - 2 \)

QUESTION 4
The diagram shows the graph of the linear relation \( y = 2x - 3 \)
Write down the graph’s gradient and the \( y \)-intercept.
The Equation of a line

INTRODUCTION

When working the previous activities you may have already discovered an important link between the equation of the linear function, \( y = ax + b \), and the gradient and vertical intercept of its graph.

As you can see here:

The line \( y = 2x - 3 \) has a gradient of 2 and a vertical intercept of \(-3\).

The line \( y = -\frac{2}{3}x + 1 \) has a gradient of \(-\frac{2}{3}\) and a vertical intercept of 1.

Notice:
- The gradient is the same as the coefficient of \( x \)
- The vertical intercept is the value of the constant in the equation.

Generally

The graph of the linear function \( y = mx + b \) has a gradient equal to the value of \( m \) and it intersects the \( y \)-axis at \( b \).

Examples to study

<table>
<thead>
<tr>
<th>Equation of graph</th>
<th>Gradient</th>
<th>( y )-intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 5x + 2 )</td>
<td>5</td>
<td>+2</td>
</tr>
<tr>
<td>( y = 0.6x - 1.5 )</td>
<td>0.6</td>
<td>-1.5</td>
</tr>
<tr>
<td>( y = -2x )</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>( y = 5 )</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>( y = 4 - \frac{1}{2}x )</td>
<td>(-\frac{1}{2})</td>
<td>4</td>
</tr>
</tbody>
</table>
Finding the equation of a line (1)

A particularly useful and important skill follows naturally from the previous section. Determine the gradient and vertical intercept of the graph of a linear function and you can write down its equation.

This technique is illustrated here.

First find the gradient by choosing two points through which the graph passes. For example: \((4, 0)\) and \((8, 2)\)

Now,

\[
\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 0}{8 - 4} = \frac{1}{2}
\]

- Second; observe that the graph intercepts the y-axis at \(-2\)
  (You now have the values of \(a\) and \(b\)).

- Finally substitute \(m = \frac{1}{2}\) and \(b = -2\) into \(y = mx + b\).

The equation of the linear function shown in figure 12 is \(y = \frac{1}{2}x - 2\).
Finding the equation of a line (2)

The previous method used to find the equation of a linear function works well if the vertical intercept is known or can be read from the graph. But, as in the following situation, this is not always the case.

An alternative technique is illustrated here.

First identify two points through which the graph passes.
For example: \((2, 6)\) and \((4, 9)\)

You can now determine the gradient of the line:

\[
\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 6}{4 - 2} = \frac{3}{2}
\]

- Second; substitute the value of this gradient, \( m = \frac{3}{2} \), and the coordinates of one of the know points, \((2, 6)\), into the general equation:

\[
6 = \frac{3}{2} \times 2 + b \\
6 = 3 + b \\
b = 3
\]

- Finally substitute \( m = \frac{3}{2} \) and \( b = 3 \) into \( y = mx + b \).

The equation of the linear function shown is \( y = \frac{3}{2}x + 3 \)
Activity 4

QUESTION 1
Write down the gradient and vertical intercept for each of the following linear functions.
(a) \( y = 6x + 2 \)  \hspace{1cm} (b) \( y = 2x - 5 \)
(c) \( y = x - 2.6 \)  \hspace{1cm} (d) \( y = 12 - x \)
(e) \( y = -\frac{5}{3}x \)  \hspace{1cm} (f) \( y = 9 \)

QUESTION 2
Referring to the graphs of the functions in question 1:
(a) Which has the steepest positive gradient?
(b) Which graph is horizontal?
(c) Which graphs slope downwards (from left to right)?
(d) Which graph passes through the origin?

QUESTION 3
Without using a table of values (ie. use the values of \( m \) and \( b \)), draw the graphs of the following:
(a) \( y = x + 1 \)  \hspace{1cm} (b) \( y = 3x - 2 \)
(c) \( y = \frac{1}{5}x + 2 \)  \hspace{1cm} (d) \( y = 10 - 5x \)
(e) \( y = 1 \)  \hspace{1cm} (f) \( y = 3x \)

QUESTION 4
Referring to the graphs of the functions in question 3:
(a) Which graphs are parallel?
(b) Which graphs are perpendicular?
(c) Which graph is horizontal?
(d) Which graph decreases as \( x \) increases?
QUESTION 5
Write down the gradient and vertical intercept for the following graph. Hence write down its equation.

QUESTION 6
Write down the gradient and vertical intercept for the following graph. Hence write down its equation.

QUESTION 7
The graph of a linear relation passes through the points (5, 8) and (10, 12). Find the graph’s equation.

QUESTION 8
Find the equation of a straight line passing through the points (20, 4) and (30, 10).
INTRODUCTION

Empirical data are pairs of measurements of two variables. For example, research into a possible relationship between adult blood-pressure and age requires a sample of adults to have their ages recorded and their blood-pressures measured. The results of these measurements could be presented in a table, such as the one below, which shows the empirical data for a sample of 10 adults.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>18</th>
<th>23</th>
<th>25</th>
<th>34</th>
<th>45</th>
<th>51</th>
<th>56</th>
<th>62</th>
<th>65</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.P. (mm.Hg)</td>
<td>115</td>
<td>116</td>
<td>118</td>
<td>123</td>
<td>120</td>
<td>122</td>
<td>125</td>
<td>126</td>
<td>130</td>
<td>138</td>
</tr>
</tbody>
</table>

For this sample of adults, analysis suggests that blood pressure increases with age. For more clarity and further analysis it's possible to show each pair of data as coordinates in the Cartesian Plane. (See below.)

Diagram showing the graphical representation of the bivariate data measuring the blood pressures and ages of a sample of 10 adults.

This type of diagram is called a scattergram.

In this example the blood pressure of adults depends upon their age. The independent variable is shown on the horizontal axis, and this is the convention when drawing scattergrams.
POSITIVE RELATIONSHIPS

Scattergrams are useful tools for analysing empirical data because they provide clear visual pictures of potential relationships between the measured variables.

In the case of the blood pressure and age it is clear that as age increases so too does blood pressure. This may not be true for you or even some people you know, but for the sample data there is certainly a tendency for blood pressure to increase with age. This is called a positive relationship. You notice that the coordinates representing the data generally rise to the right.

Another positive relationship is illustrated here. Unsurprisingly, as the number of people in a household increases, so too does the amount of water used in the shower.

NEGATIVE RELATIONSHIPS

If the scattergram shows a pattern decreasing to the right the relationship is called negative.

As one variable increases, the other decreases. An example of a negative relationship can be seen here where an athlete’s time for the 100 metres is reducing as the length of time in training increases.
NO RELATIONSHIP

Sometimes it’s difficult or impossible to discern a positive or negative relationship between two variables. The scattergram showing the Maths and Statistics examination scores for a sample of college students shows no particular upwards or downwards pattern.

Even if you believe that some skill in Mathematics is important to succeed in Statistics, there is no evidence to support this opinion in the sample data.

INDEPENDENT & DEPENDENT VARIABLES

In the earlier example showing the positive relationship between blood pressure and age – blood pressure depends upon age (rather than the other way around). In this case blood pressure is called the dependent variable. Age is the independent variable.

In the case of water usage and the number of people per household, the dependent variable is water usage. It’s not always obvious which of the variables is dependent on the other, but when it is, always use the horizontal axis to represent the independent variable.
QUESTION 1

(a) The table shows the examination scores in Maths and Science for a sample of seven students. Draw a scatter diagram to illustrate the data.

<table>
<thead>
<tr>
<th></th>
<th>Math %</th>
<th>Science %</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>20</td>
<td>18</td>
</tr>
<tr>
<td>B</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>C</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>D</td>
<td>61</td>
<td>52</td>
</tr>
<tr>
<td>E</td>
<td>80</td>
<td>60</td>
</tr>
<tr>
<td>F</td>
<td>80</td>
<td>78</td>
</tr>
<tr>
<td>G</td>
<td>90</td>
<td>70</td>
</tr>
</tbody>
</table>

(b) Describe the relationship between the two variables.

(c) Which of the two variables is the dependent variable in this case?

QUESTION 2

The table shows the heights and weights of a sample of eight students.

<table>
<thead>
<tr>
<th></th>
<th>Height (cm)</th>
<th>Weight (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>158</td>
<td>72</td>
</tr>
<tr>
<td>B</td>
<td>175</td>
<td>75</td>
</tr>
<tr>
<td>C</td>
<td>180</td>
<td>87</td>
</tr>
<tr>
<td>D</td>
<td>178</td>
<td>85</td>
</tr>
<tr>
<td>E</td>
<td>164</td>
<td>78</td>
</tr>
<tr>
<td>F</td>
<td>170</td>
<td>80</td>
</tr>
</tbody>
</table>

(a) Which is the independent variable?

(b) Draw a scattergram and comment on the relationship between height and weight.
INTRODUCTION

Did you know there is a strong relationship between the water content in certain foods and their calorific value. As the scatter diagram suggests, this relationship is negative. Foods with higher levels of water generally have fewer calories.

The strength of the relationship is indicated by the pattern in the scattergram—close to a straight line.

In situations such as these we describe the relationship as linear. A straight line drawn through the middle of the scatter is called the **line of best fit**.

A key objective of this unit is to determine the equations of lines of best fit. You will then be able to use these equations to make estimates for unknown dependent variables from independent variables. For example, in the case of food, you will be able to estimate the water content of a food containing 130 calories.
Activity 6

QUESTION 1
(a) Plot the data shown in the table and draw the line of best fit.

<table>
<thead>
<tr>
<th>X</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>30</td>
<td>45</td>
<td>58</td>
<td>90</td>
</tr>
</tbody>
</table>

(b) Find the gradient of the line of best fit.
(c) Determine the equation of the line of best fit.
(d) Find the value of \( Y \) when \( X \) is 50 assuming the linear relationship continues.

QUESTION 2
Research into how blood pressure varies with age reveals the following data:

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>20</th>
<th>28</th>
<th>35</th>
<th>50</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.P (mmHg)</td>
<td>120</td>
<td>125</td>
<td>128</td>
<td>134</td>
<td>140</td>
</tr>
</tbody>
</table>

(a) Plot the data onto the Cartesian Plane.
(b) Draw a best fit through the points.
(c) Find the gradient of the line of best fit and interpret this value.
(d) Determine the equation of the line of best fit.
(e) Estimate the blood pressure for a 45 year-old person.

QUESTION 3
An investigation into the relationship between height and weight of a sample of people revealed the following data.

<table>
<thead>
<tr>
<th>Height X (cm)</th>
<th>156</th>
<th>165</th>
<th>168</th>
<th>170</th>
<th>178</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight Y (kg)</td>
<td>65</td>
<td>67</td>
<td>71</td>
<td>75</td>
<td>84</td>
</tr>
</tbody>
</table>

(a) Plot the data onto the Cartesian Plane.
(b) Draw a line of best-fit through the points.
(c) Find the gradient of the line of best fit and interpret the units of gradient.
(d) Determine the equation of the line of best fit.
(e) Estimate the weight of a person who is 185 cm tall.
CARTESIAN PLANE
QUESTION 1 (Ref. Coordinates)

(a) A(1, 3), B(-4, -3), C(3, 0)

(b) A(1, 3), B(-4, -3), C(3, 0)

(c) (i) The abscissa of A is 1 (ii) The ordinate of B is -3

QUESTION 2 (Ref. Graphs of Linear Functions)

(a) Table of values for \( y = 3x + 2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>-7</td>
<td>-4</td>
<td>-1</td>
<td>2</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

(b) The graph of \( y = 3x + 2 \)
QUESTION 3 (Ref. The gradient of a line & its vertical intercept.)

(a) The gradient is 3. The y-intercept is 2

(b) (i) 3  (ii) −1

(c) (i) $y = −2$  (ii) $y = 2$

(d) (i) $y = x + 2$  (ii) $y = −3x + 1$

QUESTION 4 (Ref. Linear Models)

(a) (i) The cost of producing 50 scones is $120

(ii)

(iii) $C = 20$: The cost of producing zero scones is $20 (set-up cost).

(iv) The gradient is 2 indicating the cost per scone is $2.

(b) (i) $150

(ii)
(iii) The gradient is $-50$ indicating the loan-repayments are each $50$.
(iv) The value of $x$ lies between 0 and 8 inclusive.
QUESTION 5 (Ref. Lines of Best-Fit)

(a) The variable time \((t)\) is the independent variable.

(b) Graph showing the change in room temperature over a period of 20 minutes

(c) The vertical intercept is 7.5. This is the temperature of the room initially.

(d) The gradient is about 0.5. This is the temperature increase each minute.

(e) The equation of the line is \(T = 0.5t + 7.5\)

(d) When \(t = 30\) min the temperature is predicted to be about 24 degrees. This estimate assumes the linear relationship continues.

QUESTION 6 (Ref. Spreadsheets)

The data in the following table show the heights and ages of five children.

<table>
<thead>
<tr>
<th>(h) (cm)</th>
<th>95</th>
<th>110</th>
<th>130</th>
<th>140</th>
<th>156</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x) (years)</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>

(a) Height is the dependent variable in this case?

(b) (c) (d)
(e) The height of a nine-year-old is about 135cm.

(f) The rate of growth is given by the gradient; about 6.6cm per year.

(g) See diagram above.

QUESTION 7 (Ref. Simultaneous Equations)

(i) See below.

(ii) The graphs intersect at the point (2, 1)

(iii) The solutions are: $x = 2$ and $y = 1$
Solutions to Activities

ACTIVITY 1

QUESTION 1

A = (2, 3)  B = (-2, -3)  C = (-3, 0)  D = (3, -2)

QUESTION 2

• F (5, 4)
• G (-1, -3)
• H (0, -2)
• I (4, -3.5)
• J (0, -3)

QUESTION 3

(a) The ordinate of the point S is -2.

(b) The abscissa of the point T is -1.

(c) The diagram shows the points: A(2, 1), B(2, 0), C(2, -1), D(2, 4).
(d) Diagram to show the points: E(1, 3), F(−1, 3), G(0, 3), H(−2, 3).

(e) Diagram to show the points: A(−3, −2), B(−1, 0), C(0, 1), D(2, 3).

(f) Each set of points forms a pattern. The pattern is a straight line in each case.

Note:
- The pattern of points in (c) could be described by the notation $x = 2$.
- The pattern of points in (d) could be described by the notation $y = 3$.
- The pattern of points in (e) could be described by the notation $y = x + 1$.

**ACTIVITY 2**

**QUESTION 1**

(a) A table showing a sample of coordinates for the function $y = 3x − 1$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>−3</th>
<th>−2</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>−10</td>
<td>−7</td>
<td>−4</td>
<td>−1</td>
<td>2</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

(b) and (c).
QUESTION 2

(a) A sample of coordinates for the function $y = 2x - 1$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>-7</td>
<td>-5</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

(b) and (c)

(d) The graph of $y = 3x - 1$ is steeper than $y = 2x - 1$.

QUESTION 3
(a) A sample of coordinates for the function $y = 2x + 1$.

<table>
<thead>
<tr>
<th></th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>-5</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

(b) and (c).

**QUESTION 4**

(a) A sample of coordinates for the function $y = 2x - 3$.

<table>
<thead>
<tr>
<th></th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>-9</td>
<td>-7</td>
<td>-5</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

(b) and (c)

(d) They have the same steepness (are parallel) but $y = 2x - 3$ cuts the $y$-axis at $-3$ and $y = 2x + 1$ cuts the $y$-axis at $+1$. 
ACTIVITY 3

QUESTION 1

The formula for the gradient of the line is \( m = \frac{y_2 - y_1}{x_2 - x_1} \).

QUESTION 2

Find the gradient of the line joining the points \( P(5, 1) \) and \( Q(0, 4) \):

\[
m = \frac{4 - 1}{0 - 5} = \frac{3}{5}
\]

QUESTION 3

The gradient of \( AB \).

\[
m = \frac{-4 - 3}{1 - 2} = \frac{-7}{3}
\]

QUESTION 4

(a) Gradient of:

(i) \( CP = \frac{2}{7} \)

(ii) \( AD = \frac{5}{1} = 5 \)

(iii) \( BD = \frac{1}{5} \)

(b) (i) \( CD \) has a negative gradient.

(ii) \( BP \) has a positive gradient.

QUESTION 5

(a)

- Line \( FG \) where \( F = (-1, 3) \) and \( G = (-2, -1) \) has a positive gradient. (+4)
- Line \( PQ \) where \( P = (5, -2) \) and \( Q = (3, 2) \) has a negative gradient. (-2)
- Line \( ED \) where \( E = (0, 1) \) and \( D = (-3, -6) \) has a positive gradient. (7/3)

(b) The gradient is zero and therefore the line is horizontal.

(c) The gradient is infinite and therefore the line is vertical.
ACTIVITY 4

QUESTION 1
The gradients and vertical intercepts are:
(a) \( m = 6, y = 2 \)  
(b) \( m = 2, y = -5 \)  
(c) \( m = 1, y = -2.6 \)  
(d) \( m = -1, y = 12 \)  
(e) \( m = -\frac{5}{3}, y = 0 \)  
(f) \( m = 0, y = 9 \)

QUESTION 2
Referring to the graphs of the functions in question 1:
(a) \( y = 6x + 2 \) has the steepest positive gradient.  
(b) \( y = 9 \) is horizontal. It has a gradient of zero.  
(c) \( y = 12 - x \) has a negative gradient so slopes down to the right.  
(d) \( y = -\frac{5}{3}x \) passes through the origin. (The y-intercept is zero.)

QUESTION 3
Check that you have drawn graphs with:
(a) gradient = 1 and y – intercept = 1  
(b) gradient = 3 and y – intercept = -2  
(c) gradient = 1/5 and y – intercept = 2  
(d) gradient = -5 and y – intercept = 10  
(e) gradient = 0 and y – intercept = 1  
(f) gradient = 3 and y – intercept = 0

QUESTION 4
Referring to the graphs of the functions in question 3:
(a) \( y = 3x - 2 \) and \( y = 3x \) are parallel. (They have the same gradient of 3.)  
(b) \( y = \frac{1}{5}x + 2 \) and \( y = 10 - 5x \) are perpendicular. They intersect at 90 degrees.  
(c) \( y = 1 \) is horizontal  
(d) \( y = 10 - 5x \) decreases as \( x \) increases. It has a negative gradient.

QUESTION 5
gradient = 5 and y – intercept = -2 \( \Rightarrow y = 5x - 2 \)

QUESTION 6
gradient = -0.5 and y – intercept = -2 \( \Rightarrow y = -0.5x + 5 \)
**QUESTION 7**
The points are (5, 8) and (10, 12).
The graph’s gradient is:

\[ m = \frac{12 - 8}{10 - 5} \]

\[ = \frac{4}{5} \]

The gradient is 4/5 so:

\[ y = \frac{4}{5} x + b \]

We know the line passes through the point (5, 8). Substituting gives:

\[ 8 = \frac{4}{5} \times 5 + b \]

\[ 40 = 20 + 5b \]

\[ 5b = 20 \]

\[ b = 4 \]

Equation is \( y = \frac{4}{5} x + 4 \)

**QUESTION 8**
The points are (20, 4) and (30, 10).
The graph’s gradient is:

\[ m = \frac{10 - 4}{30 - 20} \]

\[ = \frac{3}{5} \]

The gradient is 3/5 so:

\[ y = \frac{3}{5} x + b \]

We know the line passes through the point (30, 10). Substituting gives:
\[ 10 = \frac{3}{5} \times 30 + b \]

\[ 50 = 90 + 5b \]

\[ 5b = -40 \]

\[ b = -8 \]

Equation is \( y = \frac{3}{5}x - 8 \)